

# Non-symmetric polarization

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Let  $P$  be an  $m$ -homogeneous polynomial in  $n$ -complex variables  $x_1, \dots, x_n$ . Clearly,  $P$  has a unique representation in the form

$$P(x) = \sum_{1 \leq j_1 \leq \dots \leq j_m \leq n} c_{(j_1, \dots, j_m)} x_{j_1} \cdots x_{j_m},$$

and the  $m$ -form

$$L_P(x^{(1)}, \dots, x^{(m)}) = \sum_{1 \leq j_1 \leq \dots \leq j_m \leq n} c_{(j_1, \dots, j_m)} x_{j_1}^{(1)} \cdots x_{j_m}^{(m)}$$

satisfies  $L_P(x, \dots, x) = P(x)$  for every  $x \in \mathbb{C}^n$ . We show that, although  $L_P$  in general is non-symmetric, for a large class of reasonable norms  $\|\cdot\|$  on  $\mathbb{C}^n$  the norm of  $L_P$  on  $(\mathbb{C}^n, \|\cdot\|)^m$  up to a logarithmic term  $(c \log n)^{m^2}$  can be estimated by the norm of  $P$  on  $(\mathbb{C}^n, \|\cdot\|)$ ; here  $c \geq 1$  denotes a universal constant. Moreover, for the  $\ell_p$ -norms  $\|\cdot\|_p$ ,  $1 \leq p < 2$  the logarithmic term in the number  $n$  of variables is even superfluous.

## 1 Introduction

It is well-known that for every  $m$ -homogeneous polynomial  $P : \mathbb{C}^n \rightarrow \mathbb{C}$  there is a unique symmetric  $m$ -linear form  $L : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$  such that  $L(x, \dots, x) = P(x)$  for all  $x \in \mathbb{C}^n$ . Uniqueness is an immediate consequence of the well-known *polarization formula* (see e.g. [6, Section 1.1]): For each  $m$ -homogeneous polynomial  $P : \mathbb{C}^n \rightarrow \mathbb{C}$  and each symmetric  $m$ -form  $L$  on  $\mathbb{C}^n$  such that  $P(x) = L(x, \dots, x)$  for every  $x \in \mathbb{C}^n$ , we have for every choice of  $x^{(1)}, \dots, x^{(m)} \in \mathbb{C}^n$

$$L(x^{(1)}, \dots, x^{(m)}) = \frac{1}{2^m m!} \sum_{\varepsilon_k = \pm 1} \varepsilon_1 \cdots \varepsilon_m P\left(\sum_{k=1}^m \varepsilon_k x^{(k)}\right).$$

Moreover, as an easy consequence, for each norm  $\|\cdot\|$  on  $\mathbb{C}^n$

$$\sup_{\|x^{(k)}\| \leq 1} |L(x^{(1)}, \dots, x^{(m)})| \leq e^m \cdot \sup_{\|x\| \leq 1} |P(x)|. \quad (1)$$

Existence can be seen as follows: Every  $m$ -homogeneous polynomial  $P : \mathbb{C}^n \rightarrow \mathbb{C}$  has a unique representation of the form

$$P(x) = \sum_{1 \leq j_1 \leq \dots \leq j_m \leq n} c_{(j_1, \dots, j_m)} x_{j_1} \cdots x_{j_m}.$$

A  $m$ -form on  $\mathbb{C}^n$  which is naturally associated to  $P$  is given by

$$L_P(x^{(1)}, \dots, x^{(m)}) := \sum_{1 \leq j_1 \leq \dots \leq j_m \leq n} c_{(j_1, \dots, j_m)} x_{j_1}^{(1)} \cdots x_{j_m}^{(m)},$$

and the symmetrization  $\mathcal{S}L_P$ , defined by

$$\mathcal{S}L_P(x^{(1)}, \dots, x^{(m)}) := \frac{1}{m!} \sum_{\sigma} L_P(x^{(\sigma(1))}, \dots, x^{(\sigma(m))}),$$

where the sum runs over all  $\sigma \in \Sigma_m$  (the set of all permutations of the first  $m$  natural numbers), then is the unique symmetric  $m$ -form satisfying  $L(x, \dots, x) = P(x)$  for every  $x \in \mathbb{C}^n$ .

Note that  $L_P$  is in general not symmetric. For an arbitrary non-symmetric multilinear form  $L : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$  and the associated polynomial  $P(x) := L(x, \dots, x)$  we have in general no estimate as in (1). Take for example  $L : (\mathbb{C}^n)^2 \rightarrow \mathbb{C}$  defined by  $(x, y) \mapsto x_1 y_2 - x_2 y_1$ . Then  $P(x) = L(x, x) = 0$ , but  $L \neq 0$ .

Our purpose is now to establish estimates as in (1) for the multilinear form  $L_P$  instead of  $\mathcal{S}L_P$ . The norms  $\|\cdot\|$  we consider on  $\mathbb{C}^n$  are 1-unconditional, i.e.  $x, y \in \mathbb{C}^n$  with  $|x_k| \leq |y_k|$  for every  $k$  implies  $\|x\| \leq \|y\|$ . Examples are the  $\ell_p$ -norms  $\|\cdot\|_p$  for  $1 \leq p \leq \infty$ .

Our main result is the following:

**Theorem 1.1.** *There exists a universal constant  $c_1 \geq 1$  such that for every  $m$ -homogeneous polynomial  $P : \mathbb{C}^n \rightarrow \mathbb{C}$  and every 1-unconditional norm  $\|\cdot\|$  on  $\mathbb{C}^n$*

$$\sup_{\|x^{(k)}\| \leq 1} |L_P(x^{(1)}, \dots, x^{(m)})| \leq (c_1 \log n)^{m^2} \cdot \sup_{\|x\| \leq 1} |P(x)|. \quad (2)$$

Moreover, if  $\|\cdot\| = \|\cdot\|_p$  for  $1 \leq p < 2$ , then there even is a constant  $c_2 = c_2(p) \geq 1$  for which

$$\sup_{\|x^{(k)}\| \leq 1} |L_P(x^{(1)}, \dots, x^{(m)})| \leq c_2^{m^2} \cdot \sup_{\|x\| \leq 1} |P(x)|. \quad (3)$$

Bearing (1) in mind, it suffices to establish the inequality

$$\sup_{\|x^{(k)}\| \leq 1} |L_P(x^{(1)}, \dots, x^{(m)})| \leq c \cdot \sup_{\|x^{(k)}\| \leq 1} |\mathcal{S}L_P(x^{(1)}, \dots, x^{(m)})|$$

with a suitable constant  $c$ . We will prove this inequality by iteration, based on the following theorem. For  $1 \leq k \leq n$  define the partial symmetrization  $\mathcal{S}_k L_P : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$  of  $L_P$  by

$$\mathcal{S}_k L_P(x^{(1)}, \dots, x^{(m)}) := \frac{1}{k!} \sum_{\sigma \in \Sigma_k} L_P(x^{(\sigma(1))}, \dots, x^{(\sigma(k))}, x^{(k+1)}, \dots, x^{(m)}).$$

**Theorem 1.2.** *There exists a universal constant  $c_1 \geq 1$  such that for every  $m$ -homogeneous polynomial  $P : \mathbb{C}^n \rightarrow \mathbb{C}$ , every 1-unconditional norm  $\|\cdot\|$  on  $\mathbb{C}^n$  and  $1 \leq k \leq m$*

$$\sup_{\|x^{(k)}\| \leq 1} |\mathcal{S}_{k-1} L_P(x^{(1)}, \dots, x^{(m)})| \leq (c_1 \log n)^k \cdot \sup_{\|x^{(k)}\| \leq 1} |\mathcal{S}_k L_P(x^{(1)}, \dots, x^{(m)})|.$$

Moreover, if  $\|\cdot\| = \|\cdot\|_p$  for  $1 \leq p < 2$ , then there even is a constant  $c_2 = c_2(p) \geq 1$  for which

$$\sup_{\|x^{(k)}\| \leq 1} |\mathcal{S}_{k-1} L_P(x^{(1)}, \dots, x^{(m)})| \leq c_2^k \cdot \sup_{\|x^{(k)}\| \leq 1} |\mathcal{S}_k L_P(x^{(1)}, \dots, x^{(m)})|.$$

The proofs require the theory of SCHUR multipliers, which was initiated by SCHUR [9]. As a crucial tool we will use norm estimates for the *main triangle projection* due to KWAPIEŃ and PEŁCZYŃSKI [7] as well as BENNETT [2] (see also [10, 11] and [3]).

## 2 Comparing coefficients

A  $m$ -linear form  $L : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$  is uniquely determined by its coefficients

$$c_{\mathbf{i}}(L) := L(e_{i_1}, \dots, e_{i_m}), \quad \mathbf{i} \in \mathcal{I}(n, m) := \{1, \dots, n\}^m,$$

where  $e_k$  denotes the  $k^{\text{th}}$  canonical basis vector in  $\mathbb{C}^n$ . With  $L_{\mathbf{i}} : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$  defined by  $(x^{(1)}, \dots, x^{(m)}) \mapsto x_{i_1}^{(1)} \dots x_{i_m}^{(m)}$  we see at once that

$$L = \sum_{\mathbf{i} \in \mathcal{I}(n, m)} c_{\mathbf{i}}(L) L_{\mathbf{i}}. \tag{4}$$

The index set  $\mathcal{I}(n, m)$  carries a natural equivalence relation:  $\mathbf{i}, \mathbf{j} \in \mathcal{I}(n, m)$  are equivalent, notation  $\mathbf{i} \sim \mathbf{j}$ , if there exists a permutation  $\sigma \in \Sigma_m$  of the first  $m$  natural

numbers such that  $i_k = j_{\sigma(k)}$  for every  $k$ . The equivalence class of  $\mathbf{i} \in \mathcal{I}(n, m)$  will be denoted by  $[\mathbf{i}]$ . It is easy to check that for every  $\mathbf{i} \in \mathcal{I}(n, m)$  there exists a unique  $\mathbf{j} \in \mathcal{J}(n, m) := \{(j_1, \dots, j_m) \in \mathcal{I}(n, m) \mid j_1 \leq j_2 \leq \dots \leq j_m\}$  such that  $[\mathbf{i}] = [\mathbf{j}]$ , respectively  $\mathbf{i} \sim \mathbf{j}$ . We will use the symbol  $\mathbf{i}^*$  to denote this unique index  $\mathbf{j}$ . For  $\mathbf{i} \in \mathcal{I}(n, m_1)$  and  $\mathbf{j} \in \mathcal{I}(n, m_2)$  we write  $(\mathbf{i}, \mathbf{j}) \in \mathcal{I}(n, m_1 + m_2)$  for the concatenation of the two.

The main idea of the proofs is now to compare  $c_{\mathbf{i}}(\mathcal{S}_k L_P)$  and  $c_{\mathbf{i}}(\mathcal{S}_{k-1} L_P)$ . For this let us compute  $c_{\mathbf{i}}(\mathcal{S}_k L_P)$ .

**Lemma 2.1.** *Let  $P : \mathbb{C}^n \rightarrow \mathbb{C}$  be an  $m$ -homogeneous polynomial and  $\mathbf{i} \in \mathcal{I}(n, m)$ . Then*

$$c_{\mathbf{i}}(\mathcal{S}_k L_P) = \frac{c_{\mathbf{i}^*}(L_P)}{|[(i_1, \dots, i_k)]|}$$

*if  $(i_{k+1}, \dots, i_m) \in \mathcal{J}(n, m - k)$  and  $\max\{i_1, \dots, i_k\} \leq i_{k+1}$ ; and otherwise*

$$c_{\mathbf{i}}(\mathcal{S}_k L_P) = 0.$$

*Proof.* By definition we have

$$\begin{aligned} c_{\mathbf{i}}(\mathcal{S}_k L_P) &= \mathcal{S}_k L_P(e_{i_1}, \dots, e_{i_m}) \\ &= \frac{1}{k!} \sum_{\sigma \in \Sigma_k} \sum_{\mathbf{j} \in \mathcal{J}(n, m)} c_{\mathbf{j}}(L_P) L_{\mathbf{j}}(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(k)}}, e_{i_{k+1}}, \dots, e_{i_m}) \\ &= \sum_{\mathbf{j} \in \mathcal{J}(n, m)} c_{\mathbf{j}}(L_P) \frac{1}{k!} \sum_{\sigma \in \Sigma_k} L_{\mathbf{j}}(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(k)}}, e_{i_{k+1}}, \dots, e_{i_m}). \end{aligned}$$

Now,  $L_{\mathbf{j}}(e_{i_{\sigma(1)}}, \dots, e_{i_{\sigma(k)}}, e_{i_{k+1}}, \dots, e_{i_m})$  equals 1 if  $\mathbf{j} = (i_{\sigma(1)}, \dots, i_{\sigma(k)}, i_{k+1}, \dots, i_m)$  and vanishes otherwise. Thus

$$c_{\mathbf{i}}(\mathcal{S}_k L_P) = \frac{c_{\mathbf{i}^*}(L_P)}{k!} \cdot |\{\sigma \in \Sigma_k \mid (i_{\sigma(1)}, \dots, i_{\sigma(k)}, i_{k+1}, \dots, i_m) \in \mathcal{J}(n, m)\}|.$$

If  $(i_{k+1}, \dots, i_m) \notin \mathcal{J}(n, m - k)$  or  $\max\{i_1, \dots, i_k\} > i_{k+1}$ , then there doesn't exist any permutation  $\sigma \in \Sigma_k$  for which  $(i_{\sigma(1)}, \dots, i_{\sigma(k)}, i_{k+1}, \dots, i_m) \in \mathcal{J}(n, m)$ . If not, then there are

$$\frac{k!}{|[(i_1, \dots, i_k)]|}$$

many permutations  $\sigma \in \Sigma_k$  for which  $i_{\sigma(1)} \leq i_{\sigma(2)} \leq \dots \leq i_{\sigma(k)}$ . □

**Proposition 2.2.** *Let  $P : \mathbb{C}^n \rightarrow \mathbb{C}$  be an  $m$ -homogeneous polynomial,  $\mathbf{i} \in \mathcal{I}(n, m)$  and  $k \in \{2, \dots, m\}$ . Then*

$$c_{\mathbf{i}}(\mathcal{S}_{k-1} L_P) = \frac{k}{|\{1 \leq u \leq k \mid i_u = i_k\}|} \cdot c_{\mathbf{i}}(\mathcal{S}_k L_P)$$

provided  $\max\{i_1, \dots, i_{k-1}\} \leq i_k$ ; and otherwise

$$c_{\mathbf{i}}(\mathcal{S}_{k-1}L_P) = 0 \cdot c_{\mathbf{i}}(\mathcal{S}_kL_P).$$

For the proof we need an additional lemma.

**Lemma 2.3.** *For every  $\mathbf{i} \in \mathcal{I}(n, k)$*

$$|[\mathbf{i}]| = |[(i_1, \dots, i_{k-1})]| \cdot \frac{k}{|\{1 \leq u \leq k \mid i_u = i_k\}|}.$$

*Proof.* Let us first examine the quantity  $|[\mathbf{i}]|$  for  $\mathbf{i} \in \mathcal{I}(n, k)$ . An easy combinatorial argument shows that

$$|[\mathbf{i}]| = \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_n!},$$

where  $\alpha_l := |\{1 \leq u \leq k \mid i_u = l\}|$ ,  $1 \leq l \leq n$ ; note that the numerator counts all permutations of the first  $k$  natural numbers and the denominator counts those permutations which give the same index.

Let now  $\beta_l := |\{1 \leq u \leq k-1 \mid i_u = l\}|$ . Then  $\alpha_l = \beta_l + 1$  for  $l = i_k$  and  $\alpha_l = \beta_l$  for all  $l \neq i_k$ . Thus

$$|[\mathbf{i}]| = \frac{k!}{\alpha_1! \alpha_2! \cdots \alpha_n!} = \frac{(k-1)!}{\beta_1! \cdots \beta_n!} \cdot \frac{k}{\alpha_{i_k}} = |[(i_1, \dots, i_{k-1})]| \cdot \frac{k}{|\{1 \leq u \leq k \mid i_u = i_k\}|}. \quad \square$$

*Proof of Proposition 2.2.* Let  $\mathbf{k} \in \mathcal{I}(n, m)$ . We decompose  $\mathbf{k} = (\mathbf{i}, l, \mathbf{j}) \in \mathcal{I}(n, m)$  with  $\mathbf{i} \in \mathcal{I}(n, k-1)$ ,  $l \in \{1, \dots, n\} = \mathcal{I}(n, 1)$ , and  $\mathbf{j} \in \mathcal{I}(n, m-k)$ . Using Lemma 2.1, the following table distinguishes three cases for the  $\mathbf{k}^{\text{th}}$  coefficient of  $\mathcal{S}_kL_P$  and  $\mathcal{S}_{k-1}L_P$ :

	$c_{\mathbf{k}}(\mathcal{S}_kL_P)$	$c_{\mathbf{k}}(\mathcal{S}_{k-1}L_P)$
(1) $\mathbf{j} \in \mathcal{J}(n, m-k)$ $l \leq j_1$ $\max\{i_1, \dots, i_{k-1}\} \leq l$	$\frac{1}{ [(\mathbf{i}, l)] } \cdot c_{\mathbf{k}^*}(L_P)$	$\frac{1}{ [\mathbf{i}]} \cdot c_{\mathbf{k}^*}(L_P)$
(2) $\mathbf{j} \in \mathcal{J}(n, m-k)$ $l \leq j_1$ $l < \max\{i_1, \dots, i_{k-1}\} \leq j_1$	$\frac{1}{ [(\mathbf{i}, l)] } \cdot c_{\mathbf{k}^*}(L_P)$	0
(3) otherwise	0	0

In case (1) we deduce by Lemma 2.3, as desired

$$\begin{aligned} c_{\mathbf{k}}(\mathcal{S}_{k-1}L_P) &= \frac{c_{\mathbf{k}^*}(L_P)}{||[\mathbf{i}]||} = \frac{||[(\mathbf{i}, l)]||}{||[\mathbf{i}]||} \cdot \frac{c_{\mathbf{k}^*}(L_P)}{||[(\mathbf{i}, l)]||} = \frac{||[(\mathbf{i}, l)]||}{||[\mathbf{i}]||} \cdot c_{\mathbf{k}}(\mathcal{S}_k L_P) \\ &= \frac{k}{|\{1 \leq u \leq k \mid i_u = l\}|} \cdot c_{\mathbf{k}}(\mathcal{S}_k L_P), \end{aligned}$$

and in the cases (2) and (3) the conclusion is evident.  $\square$

### 3 Multidimensional and classical Schur multipliers

Let  $c_{\mathbf{i}}(A)$  denote the  $\mathbf{i}^{\text{th}}$  entry of a matrix  $A \in \mathbb{C}^{\mathcal{I}(n,m)}$ . For  $A, B \in \mathbb{C}^{\mathcal{I}(n,m)}$  the ( $m$ -dimensional) SCHUR product  $A * B \in \mathbb{C}^{\mathcal{I}(n,m)}$  is defined by

$$c_{\mathbf{i}}(A * B) := c_{\mathbf{i}}(A) \cdot c_{\mathbf{i}}(B).$$

Having (4) in mind, the SCHUR product of a  $m$ -form  $L : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$  and  $A \in \mathbb{C}^{\mathcal{I}(n,m)}$  is given by

$$A * L := \sum_{\mathbf{i}} (c_{\mathbf{i}}(A) \cdot c_{\mathbf{i}}(L)) L_{\mathbf{i}}.$$

Recall that by Proposition 2.2 for each  $1 \leq k \leq m$  we have  $\mathcal{S}_{k-1}L_P = \mathfrak{A}^k * \mathcal{S}_k L_P$ , where  $\mathfrak{A}^k \in \mathbb{C}^{\mathcal{I}(n,m)}$  is defined by

$$c_{\mathbf{i}}(\mathfrak{A}^k) := \frac{k}{|\{1 \leq u \leq k \mid i_u = i_k\}|}$$

if  $\max\{i_1, \dots, i_{k-1}\} \leq i_k$ ; and  $c_{\mathbf{i}}(\mathfrak{A}^k) := 0$  otherwise. Let us decompose  $\mathfrak{A}^k$  into the SCHUR product of more handy pieces. For  $u, v \in \{1, \dots, m\}$  let  $D^{u,v} \in \mathbb{C}^{\mathcal{I}(n,m)}$  be defined by  $c_{\mathbf{i}}(D^{u,v}) := 1$  if  $i_u = i_v$  and  $c_{\mathbf{i}}(D^{u,v}) := 0$  otherwise. Define furthermore  $T^{u,v} \in \mathbb{C}^{\mathcal{I}(n,m)}$  by  $c_{\mathbf{i}}(T^{u,v}) := 1$  if  $i_u \leq i_v$  and  $c_{\mathbf{i}}(T^{u,v}) := 0$  if  $i_u > i_v$ .

With these definitions  $\mathfrak{A}^k$  decomposes as follows.

**Lemma 3.1.** *For  $1 \leq k \leq m$  we have*

$$\mathfrak{A}^k = \left( \begin{smallmatrix} k-1 \\ * \\ u=1 \end{smallmatrix} T^{u,k} \right) * \left( \sum_{u=1}^k \frac{k}{u} \cdot A^{k,u} \right) \quad (5)$$

with

$$A^{k,u} := \sum_{\substack{Q \subset \{1, \dots, k\} \\ |Q|=u}} \left( \begin{smallmatrix} * \\ q \in Q \end{smallmatrix} D^{q,k} \right) * \left( \begin{smallmatrix} * \\ q \in Q^c \end{smallmatrix} (\mathbf{1} - D^{q,k}) \right),$$

where  $Q^c$  denotes the complement of  $Q$  in  $\{1, \dots, k\}$  and  $\mathbf{1} \in \mathbb{C}^{\mathcal{I}(n,m)}$  is defined by  $c_{\mathbf{i}}(\mathbf{1}) = 1$  for all  $\mathbf{i}$ .

*Proof.* Throughout the proof, we will denote the right-hand side of (5) by  $A^k$ . Let  $\mathbf{i} \in \mathcal{I}(n, m)$ . If there exists some  $1 \leq u \leq k-1$  such that  $i_u > i_k$ , then we have by definition  $c_{\mathbf{i}}(\mathfrak{A}^k) = 0$ . On the other hand, in this case  $c_{\mathbf{i}}(T^{u,k}) = 0$  and thus  $c_{\mathbf{i}}(A^k) = 0$ .

Assume now that  $i_u \leq i_k$  for all  $1 \leq u \leq k$ . Then  $c_{\mathbf{i}}(T^{u,k}) = 1$  for all  $1 \leq u \leq k-1$ . With  $Q_{\mathbf{i}} := \{1 \leq u \leq k \mid i_u = i_k\}$  we check at once that

$$c_{\mathbf{i}}\left(\left(\bigstar_{q \in Q} D^{q,k}\right) \star \left(\bigstar_{q \in Q^c} (\mathbf{1} - D^{q,k})\right)\right) = \begin{cases} 1 & \text{if } Q = Q_{\mathbf{i}}, \\ 0 & \text{if } Q \neq Q_{\mathbf{i}}. \end{cases}$$

Therefore  $c_{\mathbf{i}}(A^{k,u})$  evaluates to 1 if  $u = |Q_{\mathbf{i}}|$  and vanishes otherwise. We have

$$c_{\mathbf{i}}(A^k) = \frac{k}{|Q_{\mathbf{i}}|} = \frac{k}{|\{1 \leq u \leq k \mid i_u = i_k\}|} = c_{\mathbf{i}}(\mathfrak{A}^k). \quad \square$$

We have seen that  $D^{u,v}$  and  $T^{u,v}$  are the building blocks of  $\mathfrak{A}^k$  under SCHUR multiplication. In what follows we will investigate the SCHUR norms of these matrices.

For a given norm  $\|\cdot\|$  on  $\mathbb{C}^n$  and  $A \in \mathbb{C}^{\mathcal{I}(n,m)}$  we denote by  $\mu_{\|\cdot\|}^m(A)$  the best constant  $c$  such that

$$\sup_{\|x^{(k)}\| \leq 1} |A * L(x^{(1)}, \dots, x^{(m)})| \leq c \cdot \sup_{\|x^{(k)}\| \leq 1} |L(x^{(1)}, \dots, x^{(m)})|$$

for any  $m$ -form  $L : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$ .

**Lemma 3.2.** *For every  $n, m$ , every  $u, v \in \{1, \dots, m\}$ , and every 1-unconditional norm  $\|\cdot\|$  on  $\mathbb{C}^n$*

$$\mu_{\|\cdot\|}^m(D^{u,v}) = 1, \quad (6)$$

$$\mu_{\|\cdot\|}^m(T^{u,v}) \leq \log_2(2n). \quad (7)$$

Moreover, for every  $1 \leq p < 2$  there exists a constant  $c_3 = c_3(p)$  so that for every  $n, m$  and  $u, v \in \{1, \dots, m\}$

$$\mu_{\|\cdot\|_p}^m(T^{u,v}) \leq c_3. \quad (8)$$

To prove this lemma we have to resort to the classical theory of SCHUR multipliers. Define  $T_n = (t_{ij}^n)_{i,j} \in \mathbb{C}^{n \times n}$  by

$$t_{ij}^n = \begin{cases} 1 & i \leq j \leq n, \\ 0 & \text{otherwise,} \end{cases}$$

and let  $I_n \in \mathbb{C}^{n \times n}$  denote the identity matrix.

**Lemma 3.3.** *We have for every  $n$*

$$\mu_{\|\cdot\|_\infty}^2(I_n) \leq 1, \quad (9)$$

$$\mu_{\|\cdot\|_\infty}^2(T_n) \leq \log_2(2n), \quad (10)$$

and, moreover, for  $1 \leq p < 2$  there is a constant  $c_3 = c_3(p)$  such that for every  $n$

$$\mu_{\|\cdot\|_p}^2(T_n) \leq c_3. \quad (11)$$

These inequalities are due to KWAPIEŃ and PEŁCZYŃSKI [7] as well as BENNETT [1]. More precisely, Proposition 1.1 of [7] gives for any matrix  $(a_{ij})_{i,j} \in \mathbb{C}^{n \times n}$

$$\sup_{\substack{\|x\|_\infty \leq 1 \\ \|y\|_\infty \leq 1}} \left| \sum_{i,j=1}^n t_{ij}^n a_{ij} y_i x_j \right| \leq \log_2(2n) \cdot \sup_{\substack{\|x\|_\infty \leq 1 \\ \|y\|_\infty \leq 1}} \left| \sum_{i,j=1}^n a_{ij} y_i x_j \right|,$$

which is (10). Statement (11) follows from Theorem 5.1 of [1], which (implicitly) states that for  $1 \leq p < 2$

$$\sup_{\substack{\|x\|_p \leq 1 \\ \|y\|_p \leq 1}} \left| \sum_{i,j=1}^n t_{ij}^n a_{ij} y_i x_j \right| \leq c_3(p) \cdot \sup_{\substack{\|x\|_p \leq 1 \\ \|y\|_p \leq 1}} \left| \sum_{i,j=1}^n a_{ij} y_i x_j \right|.$$

For the proof of (9) recall that by Theorem 4.3 of [2] and the duality  $\ell_\infty^n = (\ell_1^n)'$  we have that

$$\mu_{\|\cdot\|_\infty}^2(I_n) = \sup_{\substack{d \in \mathbb{C}^n \\ \|d\|_\infty \leq 1}} \pi_1(\ell_1^n \xrightarrow{\text{diag } d} \ell_1^n \xrightarrow{I_n} \ell_\infty^n), \quad (12)$$

where the 1-summing norm  $\pi_1$  of an operator  $T : X \rightarrow Y$  in finite dimensional spaces is defined as (see e.g. [5] or [4])

$$\pi_1(T) := \sup \left\{ \sum_{k=1}^l \|Tx_k\|_Y \mid l \in \mathbb{N}, x_k \in X, \sup_{|t_k|=1} \left\| \sum_{k=1}^l t_k x_k \right\|_X \leq 1 \right\}.$$

By the ideal property of  $\pi_1$  and the well-known fact that  $\pi_1(\ell_1^n \xrightarrow{I_n} \ell_\infty^n) = 1$  (see [8, Section 22.4.12] or [4, Section 10.4 and 11.1]) the right-hand side of (12) equals 1.

*Proof of Lemma 3.2.* We begin with the proof of (6) for the supremum norm  $\|\cdot\|_\infty$  on  $\mathbb{C}^n$ . Let  $L : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$  be a multilinear form. Without loss of generality we may



assume  $u = 1$  and  $v = 2$ . Then

$$\begin{aligned}
& \sup_{\|x^{(k)}\|_\infty \leq 1} |D^{u,v} * L(x^{(1)}, \dots, x^{(m)})| \\
&= \sup_{\|x^{(k)}\|_\infty \leq 1} \left| \sum_{\mathbf{i} \in \mathcal{I}(n,m)} d_{i_1 i_2} c_{\mathbf{i}}(L) x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} \right| \\
&= \sup_{\substack{x^{(3)}, \dots, x^{(m)} \\ \|x^{(k)}\|_\infty \leq 1}} \sup_{\substack{x^{(1)}, x^{(2)} \\ \|x^{(k)}\|_\infty \leq 1}} \left| \sum_{i,j=1}^n d_{ij} \left( \sum_{\substack{\mathbf{i} \in \mathcal{I}(n,m) \\ i_1=i \\ i_2=j}} c_{\mathbf{i}}(L) x_{i_3}^{(3)} \cdots x_{i_m}^{(m)} \right) x_i^{(1)} x_j^{(2)} \right|.
\end{aligned}$$

Using (9), we see that this is

$$\begin{aligned}
& \leq \sup_{\|x^{(k)}\|_\infty \leq 1} \left| \sum_{\mathbf{i} \in \mathcal{I}(n,m)} c_{\mathbf{i}}(L) x_{i_1}^{(1)} \cdots x_{i_m}^{(m)} \right| \\
&= \sup_{\|x^{(k)}\|_\infty \leq 1} |L(x^{(1)}, \dots, x^{(m)})|,
\end{aligned}$$

which proves  $\mu_{\|\cdot\|_\infty}^m(D^{u,v}) = 1$ . In a second step we now show that this inequality holds for any given 1-unconditional norm  $\|\cdot\|$  on  $\mathbb{C}^n$ . Again, let  $L : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$  be an  $m$ -form and fix  $x^{(1)}, \dots, x^{(m)} \in \mathbb{C}^n$  so that  $\|x^{(k)}\| \leq 1$ . With  $\tilde{L} : (\mathbb{C}^n)^m \rightarrow \mathbb{C}$  defined by

$$\tilde{L}(y^{(1)}, \dots, y^{(m)}) := L(x^{(1)} \cdot y^{(1)}, \dots, x^{(m)} \cdot y^{(m)}),$$

where  $x^{(k)} \cdot y^{(k)} := (x_1^{(k)} \cdot y_1^{(k)}, \dots, x_n^{(k)} \cdot y_n^{(k)})$ , we deduce from the first part of this proof that

$$\begin{aligned}
|D^{u,v} * L(x^{(1)}, \dots, x^{(m)})| &\leq \sup_{\substack{y^{(1)}, \dots, y^{(m)} \\ \|y^{(k)}\|_\infty \leq 1}} |D^{u,v} * \tilde{L}(y^{(1)}, \dots, y^{(m)})| \\
&\leq \sup_{\substack{y^{(1)}, \dots, y^{(m)} \\ \|y^{(k)}\|_\infty \leq 1}} |\tilde{L}(y^{(1)}, \dots, y^{(m)})| \leq \sup_{\substack{y^{(1)}, \dots, y^{(m)} \\ \|y^{(k)}\| \leq 1}} |L(y^{(1)}, \dots, y^{(m)})|;
\end{aligned}$$

note that the last inequality holds true due to the 1-unconditionality of  $\|\cdot\|$ .

The proof of (7) follows the same lines using (10) instead of (9). Finally, to prove (8) one only has to use the first step of the preceding argument with the norm  $\|\cdot\|_\infty$  replaced by  $\|\cdot\|_p$  and (9) substituted by (11).  $\square$

## 4 Proof of the Theorems 1.1 and 1.2

We are now ready to give the proofs of the Theorems 1.1 and 1.2. We begin with Theorem 1.2, as Theorem 1.1 will then follow easily.

*Proof of Theorem 1.2.* Note at first that for any 1-unconditional norm  $\|\cdot\|$  on  $\mathbb{C}^n$  the SCHUR norm  $\mu_{\|\cdot\|}^m$  turns the linear space  $\mathbb{C}^{\mathcal{I}(n,m)}$  into an BANACH algebra. By Lemma 3.1 and (6),

$$\begin{aligned} \mu_{\|\cdot\|}^m(A^{u,k}) &\leq \sum_{\substack{Q \subset \{1,\dots,k\} \\ |Q|=u}} \left( \prod_{q \in Q} \mu_{\|\cdot\|}^m(D^{q,k}) \right) \cdot \left( \prod_{q \in Q^c} \underbrace{\mu_{\|\cdot\|}^m(\mathbf{1} - D^{q,k})}_{\leq 2} \right) \\ &\leq \sum_{\substack{Q \subset \{1,\dots,k\} \\ |Q|=u}} 1^{|Q|} 2^{|Q^c|} = \binom{k}{u} 2^{k-u}, \end{aligned}$$

and thus

$$\begin{aligned} \mu_{\|\cdot\|}^m(\mathfrak{A}^k) &\leq \mu_{\|\cdot\|}^m \left( \bigast_{u=1}^{k-1} T^{u,k} \right) \cdot \mu_{\|\cdot\|}^m \left( \sum_{u=1}^k \frac{k}{u} \cdot A^{k,u} \right) \\ &\leq (\mu_{\|\cdot\|}^m(T^{u,k}))^{k-1} \cdot k \sum_{u=1}^k \binom{k}{u} 2^{k-u} \leq k 3^k (\mu_{\|\cdot\|}^m(T^{u,k}))^{k-1}. \end{aligned}$$

Finally, the results in (7) and (8) complete the proof.  $\square$

We remark that the best constants  $c_1$  and  $c_2$  in Theorem 1.2 satisfy the estimates  $(c_1 \log n)^k \leq k 3^k (\log_2(2n))^{k-1}$  and  $c_2^k \leq k 3^k c_3^{k-1}$  with  $c_3$  denoting the constant in (8).

We finish with the proof of our main theorem.

*Proof of Theorem 1.1.* Repeated application of Theorem 1.2 yields

$$\begin{aligned} &\sup_{\|x^{(k)}\| \leq 1} |L_P(x^{(1)}, \dots, x^{(m)})| \\ &= \sup_{\|x^{(k)}\| \leq 1} |\mathcal{S}_1 L_P(x^{(1)}, \dots, x^{(m)})| \\ &\leq c^2 \cdot \sup_{\|x^{(k)}\| \leq 1} |\mathcal{S}_2 L_P(x^{(1)}, \dots, x^{(m)})| \\ &\leq \dots \\ &\leq c^{2+\dots+(m-1)+m} \cdot \sup_{\|x^{(k)}\| \leq 1} |\mathcal{S}_m L_P(x^{(1)}, \dots, x^{(m)})|, \end{aligned}$$

with  $c$  denoting the respective constants of Theorem 1.2. Finally, (1) (which is an immediate consequence of the polarization formula) completes the argument (note that by definition  $\mathcal{S}L_P = \mathcal{S}_m L_P$ ).  $\square$

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